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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE PUBLISHED AND UNPUBLISHED WORK OF CHARLES STURM ON ALGEBRAIC AND DIFFERENTIAL EQUATIONS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, APRIL 28, 1911.

BY PRESIDENT MAXIME BÔCHER.

CHARLES Sturm was born in 1803 at Geneva, then a part of France, and went to Paris at about the age of twenty-one. There he spent the rest of his life and died in 1855, having become a member of the French Academy of Sciences in 1836.

It is not necessary for us to go beyond this bare outline of Sturm's life, since it is not with his worldly fortunes that we shall be concerned. Neither do I propose to give an account of his life-work as a whole.* The brief biography and bibliography prefixed to his posthumous Cours d'Analyse fulfills to some extent both of these purposes. We shall confine ourselves to one branch of investigation pursued by Sturm: the study of the real solutions of algebraic equations and of linear differential equations, both ordinary and partial. It was here that Sturm's most important and suggestive work was done, and it is of interest to try to gain some insight into the relations between the various parts of the subject as they appeared to him.

The papers with which we are concerned may be exhibited in the following table:

* In brief we may say that, besides the investigations with which we shall be concerned, Sturm published

(*a*) An experimental memoir in collaboration with Colladon on the compressibility of liquids.

(*b*) A large number of minor papers, mostly geometrical.

(*c*) Several papers on geometrical optics including a long memoir.

(*d*) Some papers, partly in collaboration with Liouville, on the imaginary roots of equations, which are not without connection with Sturm's work on the real roots of algebraic equations.

THE PAPERS OF 1829, presented to the Academy on the dates given and summarised, as indicated, in the *Bulletin de Férussac:*

May 23. "Sur la résolution des équations numériques." Volume 11, pages 419–422, and volume 12, page 318, footnote.

June 1. No title given. Volume 11, pages 422–424, and volume 12, page 318, footnote. The subject of this memoir is the equation $Ax^{\alpha} + Bx^{\beta} + \cdots + Mx^{\mu} = 0$, where $\alpha, \beta, \ldots, \mu$ are real but not necessarily rational.

June 8. Note. Volume 11, page 425. It is merely stated that this note contains (1) two new proofs of the reality of the roots of the transcendental equations to which the solution of various problems in mathematical physics leads; (2) the general determination of the constant coefficients in the series for representing an arbitrary function between given limits.

July 27. "Sur l'intégration d'un système d'équations différentielles linéaires." Volume 12, pages 314–322.

August 3. "Sur la distribution de la chaleur dans un assemblage de vases." Volume 12, page 322. Nothing but the title of this paper is preserved.

October 19. "Nouvelle théorie relative à une classe de fonctions transcendantes que l'on rencontre dans la résolution des problèmes de la physique mathématique." Volume 12, page 322. Nothing but the title of this paper is preserved unless, as is possible, a brief statement in volume 11, pages 424–425, refers to it.

THE THREE GREAT MEMOIRS.

1835. "Mémoire sur la résolution des equations numériques." *Mémoires des savants étrangers*, volume 6, pages 271–318.

1836. "Mémoire sur les équations différentielles linéaires du second ordre." *Liouville's Journal*, volume 1, pages 106–186. This memoir had been presented to the Academy September 30, 1833, and an abstract of it published in *l'Institut* for November 9, 1833, pages 219–223.

1836. "Mémoire sur une classe d'équations à différences partielles." *Liouville's Journal*, volume 1, pages 373–444. Cf. *l'Institut* for November 30, 1833, pages 247–248.

For the sake of completeness we note that there are also three minor papers of later date.*

It is in the three great memoirs of 1835–36 that Sturm gave its final form to so much of his work as he completed; but the above list strongly suggests, what a closer study amply confirms, that it was in the year 1829 that the great creative period of Sturm's life fell, and that the papers presented to the Academy in that year, so far as they are still accessible, must be examined if we would gain an insight into the lines of thought followed by him in making his great discoveries. In doing this we shall find that certain not uninteresting aspects of his early work find little or no mention in the great memoirs. Most of this early work is preserved to us only in the form of brief abstracts, sometimes even only by ts title, so that some reconstruction becomes necessary. This makes it impossible for us to attain certainty at all points, but perhaps the discussion is not less interesting for this reason.

Sturm's personal and scientific relations to Fourier form an indispensable background to a consideration of the papers presented to the Academy in such rapid succession during the summer of that fruitful year. The two main subjects of Fourier's life work had been the theory of heat and the theory of the solutions of numerical equations. Both of these subjects were carried forward by Sturm, the first in the two memoirs of 1836, the second in that of 1835. But if in the memoirs these tendencies appear quite distinct, we find them, when we turn to the papers of 1829, blended in a most curious and interesting manner.

Fourier's treatise on the solution of numerical equations was not published until 1831, after the author's death; but the manuscript of this work had already in 1829 been communicated to several persons among whom was Sturm, who tells us explicitly in the paper of May 23 what a strong influence it had on his own work.

Fourier had established the theorem that a real algebraic equation of the kth degree, $f(x) = 0$, cannot have more roots in an interval ab, neither of whose extremities is a root, than the difference between the number of variations of sign in the

* Namely: a brief extract of a memoir written by Sturm and Liouville together: "Sur le développement des fonctions en séries . . . ," published in *Liouville's Journal*, vol. 2 (1837), pp. 220–223, and also in the *C. R.*, vol. 4, p. 675; and two papers concerning the real roots of algebraic equations in *Liouville's Journal*, vol. 7 (1842), pp. 132–133, 356–368.

set of functions

$$f(x),\ f'(x),\ f''(x),\ \ldots,\ f^{[k]}(x), \tag{1}$$

(accents denoting differentiation) at the points a and b. Sturm's theorem, as it is still called, replaces the sequence (1) by

$$f_0(x),\ f_1(x),\ f_2(x),\ \ldots, \tag{2}$$

which coincides with (1) in the first two places, while each subsequent f_n is the negative of the remainder obtained by dividing f_{n-2} by f_{n-1},

$$f_{n-2}(x) = q_{n-1}(x)f_{n-1}(x) - f_n(x) \quad (n = 2, 3, \ldots). \tag{3}$$

The advantage of this set over the set (1) is that the difference in the number of variations in (2) at a and b is *precisely equal* to the number of roots between a and b. Since this theorem is given in the first of the notes of 1829 and is elaborated at length in the first of the great memoirs, one might be tempted to suppose that this formed the starting point in Sturm's researches. Fortunately Sturm himself has preserved us from this mistake, for on the closing page of the first memoir of 1836 he tells us that the above theorem was merely a by-product of his extensive investigations on the subject of linear difference equations of the second order. Curiously enough, however, this subject of difference equations is nowhere else alluded to in Sturm's published writings.

The key to this difficulty lies, I feel sure, in the paper of August 3, 1829, of which as has been said only the title is preserved. This memoir is described as more extensive than the one of July 27, which, as one sees from the summary, was not brief. At Sturm's death there was found among his papers a "very extensive" memoir with almost precisely the same title,* which has also never been published. I shall try to show you how this lost paper forms the starting point in Sturm's investigations, and how all his other work which concerns us here grew directly out of it. For this purpose we must first reconstruct at least the general framework of this paper.

* "Sur la communication de la chaleur dans une suite de vases." Cf. Cours d'Analyse, vol. 1, p. xxviii. It is there said: "Ces deux mémoires" (*i. e.*, this one and one on curves of the second order) "sont en état d'être imprimés, et M. Liouville a bien voulu se charger de leur publication." It is to be regretted that this intention was never carried out. Even at this late date the publication would be decidedly interesting if by chance the manuscript could still be found.

Suppose we have a number of vases $P_0, P_1, \ldots, P_n$ placed in any position with reference to one another and filled with various liquids at diverse temperatures. These vases we suppose to be immersed in an atmosphere which circulates freely and thus maintains a constant temperature which we take as the zero of our scale. Each of these vases radiates heat into this atmosphere, and the vases also interchange heat among themselves by radiation. Let us denote the temperature of the vase P_i at the time t by $u\ (i, t)$. The differential equation for the flow of heat is then, if we assume the Newtonian law of radiation,

$$c(i)\frac{\partial u(i,t)}{\partial t} = k(i,0)u(0,t) + \cdots + k(i,n)u(n,t) \qquad (i = 0, 1, \ldots, n). \tag{4}$$

Here $c(i)$ is a positive constant depending on the specific heat of the vase P_i; when $i \neq j$, $k(i, j) = k(j, i)$ is a positive constant of proportionality which measures the amount of radiation between P_i and P_j; and finally $k(i, i)$ is written merely as an abbreviation for

$$k(i,i) = -h(i) - k(i,0) - \cdots - k(i,i-1) - k(i,i+1) - \cdots - k(i,n),$$

where $h(i)$ is a positive constant of proportionality for measuring the radiation of P_i into the atmosphere. It is important for us to understand that the constant $k(i, j)$, when $i \neq j$, depends for its value not merely on the relative positions and the sizes and shapes of the vases P_i and P_j, but that its value is also decreased if one or more of the other vases is so placed as to cut off part of the radiation from P_i to P_j. If these vases are *completely* cut off from each other by intervening vases, the constant $k\ (i, j)$ has the value zero.

I suspect that it was precisely to the problem just indicated and in particular to the system of equations (4) that Sturm first turned his attention. There is, however, evidence (cf. the paper of July 27) that he had also under consideration problems in small vibrations and in celestial mechanics leading to systems of differential equations analogous to (4) but which may be somewhat more general in form. The paper last cited was devoted to the analytical treatment of such systems of linear

homogeneous differential equations with constant coefficients. As the chief result is a method of treating the algebraic *characteristic equation* of the system, we postpone any discussion of this paper until later.

It is not, however, in more general, but rather in more special problems that Sturm found his real inspiration. Consider the case in which the vases $P_0, \ldots, P_n$ are arranged in linear sequence* and in such a way that the radiation between two non-consecutive vases is completely cut off by the intervening ones. This case is characterised analytically by all the constants $k(i, j)$ vanishing except those for which the integers i, j are either equal or differ from one another by unity. We may then write equations (4) as follows:

$$
\begin{aligned}
c(0)\frac{\partial u(0,t)}{\partial t} &= -[h(0)+k(0,1)]u(0,t)+k(0,1)u(1,t),\\
c(i)\frac{\partial u(i,t)}{\partial t} &= k(i,i-1)u(i-1,t)-[h(i)+k(i,i-1)\\
&\quad +k(i,i+1)]u(i,t)+k(i,i+1)u(i+1,t)\\
&\qquad (i=1,2,\ldots,n-1),\\
c(n)\frac{\partial u(n,t)}{\partial t} &= k(n,n-1)u(n-1,t)\\
&\quad -[h(n)+k(n,n-1)]u(n,t).
\end{aligned}
\tag{5}
$$

By the side of the problem in the theory of heat which we have just formulated we may advantageously consider the problem of the small transverse vibrations of a stretched elastic string whose mass is negligible but which is weighted at a number of points by heavy particles. This problem had first been considered one hundred years earlier by John Bernoulli, and for half a century this and equivalent problems had been, in more or less general forms, subjects of investigation by Daniel Bernoulli, Euler, and Lagrange;† but none of these mathematicians had gone beyond the case where the particles

* It is interesting to note that in the title of the manuscript found at the time of Sturm's death, and which may be supposed to be the final form which his memoir took, the phrase "une suite de vases" is used in place of the earlier "un assemblage de vases." This change suggests that as his work developed Sturm desired to give more prominence to this special case; or, indeed, he may have eliminated all consideration of the more general case.

† Cf. Burkhardt's Report in the *Jahresbericht d. deutschen Mathematiker-Vereinigung*, vol. 10 (1901–1908).

are of equal mass and are equally spaced,* and for fifty years the problem had practically remained untouched.

Suppose that the string in its position of equilibrium lies along the axis of x and that particles $P_0, \ldots, P_n$ with masses $c(0), \ldots, c(n)$ lie respectively at the points whose abscissas are $x_0 < x_1 < \cdots < x_n$, and let P_0 and P_n be at the ends of the string. In order to secure the same degree of generality as in the heat problem above mentioned, we assume that during the transverse vibration each particle is drawn back towards its position of equilibrium not merely by the tension of the string but also by an additional force acting towards its position of equilibrium and proportional to the distance from this point. The constant of proportionality here we denote by $h(i)$ in the case of the ith particle, and, calling the tension of the string T, we let

$$k(i, i-1) = \frac{T}{x_i - x_{i-1}}.$$

The distance of P_i from its position of equilibrium at the time t we denote by $u(i, t)$, and we assume that each particle is free to move only in a direction at right angles to the axis of x, and that the whole motion takes place in one plane. Finally we assume that the string always remains so nearly straight that the squares of the sines or tangents of the angles which its pieces make with the axis of x may be neglected.† Then it is readily seen that the equations of motion of the particles become identical with equations (5) *provided we replace the first derivatives in these equations by second derivatives.* It follows that the mechanical problem last mentioned is mathematically almost equivalent to the problem in the theory of heat considered above. That Sturm chose the latter rather than the former, with which he was surely familiar, is, so far as the greater part of his work goes,‡ a matter of slight importance and is prob-

* In all the cases treated during the eighteenth century the particles are either supposed to be acted upon by no external forces, or to be under the influence of gravity acting in the direction of the string. The presence of such an external constant force as this is mathematically equivalent to an unequal spacing of the particles, and will therefore not be explicitly considered by us.

† It should be noticed that we do not assume the ends of the string to be fixed. If we did this, we should have a case strictly analogous to that in which the two extreme vases, P_0 and P_n, in the heat problem are maintained at the temperature zero.

‡ An essential difference occurs only in the work which leads up to the latter part of the second memoir of 1836.

ably to be explained by the fact that the theory of heat was at that moment a more "up to date" subject. The relation between the two problems is however so close that we may be permitted to depart so far from strict historical accuracy as to substitute the vibration problem for the heat problem in our further explanations, since in this way greater concreteness of expression may be gained.

We consider first the simple harmonic vibrations of the string, that is we assume that u has the form

$$u(i, t) = y(i)\,[A \sin \mu t + B \cos \mu t]. \tag{6}$$

By substituting this expression in the equations of motion we find for $y(i)$ the difference equation of the second order

$$\begin{aligned} k(i, i-1)y(i-1) - [h(i) + k(i, i-1) + k(i, i+1) \\ - \mu^2 c(i)]\, y(i) + k(i, i+1)y(i+1) = 0 \end{aligned} \tag{7}$$

or

$$\Delta\{k(i, i-1)\Delta y(i-1)\} + [\mu^2 c(i) - h(i)]y(i) = 0 \tag{7'}$$

together with the terminal conditions

$$k(1, 0)y(1) + [\mu^2 c(0) - h(0) - k(1, 0)]y(0) = 0, \tag{8}$$

$$[\mu^2 c(n) - h(n) - k(n, n-1)]y(n) + k(n, n-1)y(n-1) = 0. \tag{9}$$

The equation (7) has in general no solution other than zero which satisfies both conditions (8) and (9),—it is only for special values of μ that these conditions can both be fulfilled. Consequently we shall disregard at first condition (9), and consider merely the solution of (7) which satisfies (8). This solution obviously contains an arbitrary constant factor, since (7) and (8) are homogeneous. We therefore replace the condition (8) by the two non-homogeneous conditions

$$\begin{aligned} y(0) &= k(0, 1), \\ y(1) &= h(0) + k(0, 1) - \mu^2 c(0), \end{aligned} \tag{8'}$$

which are precisely equivalent to (8) except that (8') also determines the otherwise arbitrary constant factor, and determines it in such a way that $y(i)$ does not vanish identically.

The solution $y(i)$ determined by (8') does not in general satisfy

(9). It does, however, for a specified value of μ either satisfy the condition $y(n) = 0$, or a condition of precisely the form (9) where either $c(n)$ or $h(n)$ have in general been replaced by another value.* We may therefore say that for every value of μ the function $u(i, t)$ defined by (6) gives a simple harmonic vibration corresponding either to the mechanical problem we wish to consider or to a modification of this problem which consists either in having the particle P_n held fast, or in a change in the mass of this particle, or in a change in the strength of the force which pulls this particle back to its position of equilibrium. This we shall speak of as the modified problem corresponding to a given value of μ, using this term so that, for the special values of μ above referred to, the original problem itself is the modified problem.

In the plane in which the vibration takes place let us now construct an ordinate of length $y(i)$ at each of the points x_i and connect the extremities of each two successive ordinates by a straight line. The broken line thus formed, which we shall call the line $y(i)$, gives essentially the shape of the string in the simple harmonic vibration we are considering; for, if we multiply by $A \sin \mu t + B \cos \mu t$ all the ordinates of this broken line, and this will evidently not essentially affect its shape, we get a broken line which has precisely the shape of the string at the time t. The points where the line $y(i)$ meets the axis of x thus give the *nodes* of the simple harmonic vibration in question. Either from simple mechanical considerations or from the equation (7) we see that at each node the line $y(i)$ crosses the axis. Consequently, since each of the quantities $y(0)$, $y(1)$, $\cdots$, $y(n)$ is obviously a continuous function of μ (in fact a polynomial in μ^2), the nodes also vary continuously with μ, never suddenly appearing or disappearing except at the extremity x_n of the string.

There can be very little doubt that at this point Sturm, by a simple manipulation of equation (7′) which we will not stop to give here†, established the important fact, which may easily

* This is true even in the case $\mu = 0$ provided we replace $h(n)$ by a negative quantity. In all other cases positive quantities may be used in place of $c(n)$ and $h(n)$.

† Cf. Porter, *Annals of Mathematics*, second series, vol. 3 (1902), p. 55. In the article here cited Professor Porter, at my suggestion, reconstructed a part of Sturm's researches on difference equations without, however, considering either the vibration problem or the heat problem. Cf. also the article by Professor Porter's pupil, Miss Merrill, *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 423.

have been suggested to him by the mechanical problem itself, that as μ^2 increases the abscissa of each node decreases, new nodes appearing one by one at the point x_n.

It is here that Sturm must have noticed the connection with Fourier's theorem concerning the roots of algebraic equations. To establish this connection we need merely to observe that the number of nodes for a given value of μ^2 (we shall write for convenience $\lambda = -\mu^2$) is simply the number of variations of sign in the set of polynomials in λ

$$y(n), y(n-1), \cdots, y(0). \tag{10}$$

Consequently what we have said above is equivalent to saying that the number of roots of the polynomial $y(n)$ between two negative values of λ is precisely equal to the difference between the number of variations in the system (10) for these two values of λ. Sturm found, therefore, that for the particular polynomial $y(n)$ he was in possession of a sequence of polynomials of descending degrees (since $y(i)$ is a polynomial of the ith degree in λ), which served *perfectly* the purpose which the sequence of derivatives serves imperfectly in Fourier's theorem. He must then have asked himself to what properties of the polynomials (10) this fact is due, and have seen that just three properties were used:

(*a*) The last polynomial, $y(0)$, is a constant not zero.

(*b*) When one polynomial vanishes the two adjacent ones have opposite signs. This was an immediate consequence of (7), but would follow in the same way if instead of (7) the y's satisfied any difference equation

$$L(i)y(i+1) + M(i)y(i) + N(i)y(i-1) = 0, \tag{11}$$

where L and N always have the same sign.

(*c*) The nodes increase with λ. Far less than this, however, would be sufficient for our present purpose, namely that when a node lies at x_n it decrease as λ decreases. This would in particular be the case if $y(n)$ had no multiple root and $y(n-1)$ were simply the derivative of $y(n)$, for this is precisely the property of the derivative on which Fourier's work is based.

We may then suppose that Sturm said to himself: Starting with any polynomial without multiple roots, $f(\lambda)$, in place of $y(n)$, and taking in place of $y(n-1)$ the derivative $f'(\lambda)$,

how can I form a sequence of polynomials $f_2(\lambda), f_3(\lambda), \ldots,$ to take the place of $y(n-2), y(n-3), \ldots,$ which satisfy a relation of the form (11) and of which the last is a constant not zero? This question once formulated, the method of successive divisions and reversal of sign of the remainder, leading to equation (3) which is merely a special case of (11), would readily suggest itself, and Sturm's theorem in its most familiar form was found.

If our surmises so far are correct, it follows that, even at this early date Sturm must have been well aware that any sequence of functions having a small number of easily specified properties would serve the purpose of his theorem just as well as the sequence (2); so that to call such more general sequences Sturmian sequences, as is now done,* is even from a strictly historical point of view entirely suitable. Our belief that Sturm was familiar with this more general point of view need not, however, rest entirely on the line of reasoning so far explained. Not only does he show in his memoir of 1835 how other Sturmian sequences besides (2) may be formed;† but more particularly his paper of July 27, 1829, to which reference has already been made, is mainlv devoted to the formation for a special algebraic equation of a Sturmian sequence which is very different from the sequence (2). If we use the notation of determinants, which Sturm does not use, the equation in question is

$$(12) \qquad \begin{vmatrix} g_{11}\lambda + k_{11} & g_{12}\lambda + k_{12} & \cdots & g_{1n}\lambda + k_{1n} \\ g_{21}\lambda + k_{21} & g_{22}\lambda + k_{22} & \cdots & g_{2n}\lambda + k_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ g_{n1}\lambda + k_{n1} & g_{n2}\lambda + k_{n2} & \cdots & g_{nn}\lambda + k_{nn} \end{vmatrix} = 0,$$

where $g_{ij} = g_{ji}, k_{ij} = k_{ji},$ and where the g's and k's are real and the former are the coefficients of a non-singular definite quadratic form. Sturm falls here into the error into which Laplace and Lagrange had fallen before him, and which was first corrected by Weierstrass in 1858, of thinking that this equation can have no multiple roots. He gives, however, the correct theorem that the roots are all real; and, what is important for us here, he states that the determinant in (12) and the polynomials obtained by striking off from it the last 1, 2, 3, $\cdots$ rows and columns form

* Cf. Weber's Algebra, 2 ed., vol. 1, p. 303.

† Cf. also the closing lines of his abstract of May 23, 1829.

a Sturmian sequence,* provided each of these polynomials is multiplied by such a power of -1 as to make the coefficient of its leading term positive.†

The paper of May 23, 1829, subsequently published as the memoir of 1835, did more than all of his other papers together to win for its author recognition both in France and throughout Europe. It appears to us here in its true light,‡ as a digression from his investigations in the domain of mechanics and mathematical physics. This digression was indeed carried a little farther, as the paper of June 1 testifies. Here it is shown how Fourier's methods can be applied with very slight change to obtain an upper limit for the number of real roots in a given interval for the type of transcendental equation there considered. It is clear, moreover, from a brief remark near the middle of page 424 that Sturm was here also in possession of a method of forming a Sturmian sequence.

Let us now return to the problem of the vibrating string which we were considering above. We saw that as μ^2 increases the abscissa of each node decreases. Now from (8′) and (7′) it may readily be inferred that when $\mu = 0$,

$$0 < y(1) < y(2) < \cdots < y(n),$$

so that here there is no node. On the other hand from (8′) and (7) we see that $y(i)$ is a polynomial of the ith degree in μ^2 whose leading coefficient has the sign of $(-1)^i$. Consequently, for very large values of μ^2 we have a node in each of the intervals

$$x_i < x < x_{i+1}, \qquad (i = 0, 1, \cdots, n),$$

that is, we have the maximum possible number of nodes, namely n. Accordingly as μ^2, starting with the value zero, increases, we have at first no node, then for a while one node,

* This is true only with the qualification that no two of these polynomials have a common root; cf. Weber's Algebra, 2 ed., vol. 1, p. 308. This necessary qualification is not mentioned in Sturm's abstract, though it is by no means impossible that it may have been given in the extended memoir.

† In other words, these factors are all $+1$ or alternately $+1$ and -1 according as the quadratic form of which the g's are the coefficients is positive or negative.

‡ There are further details elaborated in this memoir to which it is not necessary for us to refer.

then for a certain interval two nodes, etc., until finally the nth node appears at x_n and from that point on we constantly have n nodes. We thus see that there are just n positive values of μ^2, for which a node lies at x_n, that is, that the equation in μ^2, $y(n) = 0$, has n distinct positive roots, and consequently, since it is of the nth degree, that it has no imaginary or negative or multiple roots. If we denote the roots arranged in order of increasing magnitude by $\mu'^2, \mu''^2, \cdots, \mu^{[n]2}$ it is clear from what has been said that for positive values of μ^2 $y(i)$ has just k nodes ($k \leqq n$) in the interval $x_0 < x < x_n$ when and only when $\mu^{[k]2} < \mu^2 < \mu^{[k+1]2}$, where, for convenience, we let $\mu^{[0]} = 0$, $\mu^{[n+1]} = +\infty$.

The next step is to show that in this interval there exists one and only one value of μ^2 for which $y(i)$ satisfies the condition (9), and this follows readily from the fact* that, when $y(n) \neq 0$, $y(n-1)/y(n)$ increases as μ^2 increases, and hence increases from $-\infty$ to $+\infty$ as μ^2 increases through the interval we are now considering. This establishes the following:

THEOREM OF OSCILLATION.† There exist just $n+1$ values of μ^2, all real and positive, for which the difference equation (7) has a solution, not identically zero, which satisfies the terminal conditions (8), (9). Denoting these values, arranged in order of increasing magnitude, by $\mu_0^2, \mu_1^2, \cdots, \mu_n^2$, the solution $y_k(i)$ of (7) corresponding to μ_k^2 and satisfying (8), (9) has exactly k nodes.

We obtain in this way $n+1$ simple harmonic vibrations of the weighted string, which differ from one another in the number of their nodes. The most general motion of the string will be obtained by compounding these simple harmonic vibrations. Here the formulæ, which are readily obtained, for making the vibration correspond to arbitrarily given initial conditions are closely analogous to the well known ones for the representation of periodic functions at n equally spaced points by a finite trigonometric series. These latter formulæ had been obtained by Lagrange in considering a special case of the vibration problem with which we have been concerned.

In precisely the same way, the problem of the distribution of

* Cf. Porter, loc. cit. This fact is also necessary in the proof, referred to above, that the nodes decrease as μ^2 increases.

† See the articles by Porter and Miss Merrill above cited. This theorem, never published by Sturm, is the prototype of the important and still increasing class of theorems to which Klein attached this name more than fifty years later.

heat in a row of vases is solved by the formula

$$(13) \qquad A_0 e^{-\lambda_0 t} y_0(i) + A_1 e^{-\lambda_1 t} y_1(i) + \cdots + A_n e^{-\lambda_n t} y_n(i),$$

where $\lambda_k = -\mu_k^2$. Here also the coefficients A_k can be determined so as to correspond with given initial conditions. It was, however, not merely the analytic solution of this problem which interested Sturm, but even more, perhaps, a discussion of the properties of the solution.

If we mark the vases at any moment + or − according as their temperatures are above or below the temperature of the surrounding atmosphere, it is obvious from physical considerations that as time passes the number of variations of sign in the sequence cannot increase. This fact may be established analytically by means of the equations (5) of which the function (13) is a solution, and at the same time we can see precisely how the number of variations decreases. For a fixed value of t we plot the function (13) as a broken line, precisely as above we plotted the line $y(i)$. This line, however, unlike the line $y(i)$, does not necessarily cross the axis of x at every point where it meets it; indeed it may meet the axis of x not merely at isolated points but it may also coincide with it throughout a whole segment extending between two points x_i. Let us call each of the isolated points and segments where the broken line representing (13) meets the axis of x a node of (13). To each node we attribute a multiplicity as follows: If the node lies at, or reaches up to, one of the end-points x_0 or x_n, we take as its multiplicity the number of points x_i contained in it. Otherwise we take either this number or a number one greater, in such a way that the multiplicity shall be odd or even according as in passing through the node the function (13) does or does not change sign. This convention is justified by the fact, readily established by means of equations (5), that such multiple nodes can occur only for isolated values of t; and that for values of t a little smaller than such a value, the function (13) has exactly k simple nodes in the neighborhood of the point or segment where a k-fold node is to appear.* Now the fundamental fact here, which also follows from (5), is that as t increases through a value for which there is a node of multiplicity k, the simple nodes, after coalescing to form the multiple node, all disappear,

* We exclude here and in what follows the possibility that the function (13) vanish identically; or, what is the same thing, we assume that not all the A's are zero.

leaving no node at all in the neighborhood of the point or segment in question, except in the one case of a node of odd multiplicity which does not lie at or reach up to one of the endpoints x_0 or x_n; in which case just one simple node remains in the neighborhood in question.

From these considerations it may readily be inferred that the number of nodes of the expression

$$A_p y_p(i) + A_{p+1} y_{p+1}(i) + \cdots + A_q y_q(i), \tag{14}$$

where $A_p \neq 0$, $A_q \neq 0$, $0 \leqq p < q \leqq n$, cannot be less than p or greater than q, a multiple node being counted at pleasure either once or as often as its multiplicity indicates. For if, by introducing exponentials, we modify (14) into an expression of the form (13), the nodes of (14) appear merely as the nodes of (13) when $t = 0$, and the number of such nodes lies, by what was said above, between the number of nodes of the expression (13) in question for a very large negative, and the number for a very large positive value of t. For such extreme values, (13) coincides very nearly with constant multiples of $y_p(i)$ and $y_q(i)$ respectively.

While it does not seem likely from the scanty evidence which Sturm has left us that these latter considerations were all familiar to him in 1829, they can hardly fail to have been in his possession four years later, and it is not unlikely that if the manuscript which was among his papers at the time of his death could be recovered, it would be found to contain a systematic exposition of them along with the other matters we have touched upon.*

Two lines for further investigation now naturally presented themselves. One of these consisted in replacing the difference equation (7) and the boundary conditions (8), (9) by the more general relations of the same form

$$\begin{gathered} \Delta\{K(i)\Delta y(i)\} - G(i)y(i+1) = 0 \quad (K(i) > 0), \\ K(0)\Delta y(0) - hy(0) = 0 \text{ (or } y(0) = 0), \\ K(n-1)\Delta y(n-1) + Hy(n-1) = 0 \text{ (or } y(n) = 0). \end{gathered} \tag{15}$$

* We note in passing that if we equate the expression (13) to zero and assign to i a particular value, the equation thus obtained is precisely of the form considered in the paper of June 1, 1829 concerning which we have already spoken. Cf. the introductory remarks in the summary of that paper.

If here we assume that the quantities $K(i)$, $G(i)$, h, H are continuous increasing functions of λ, the results relating to equation (7) and also their proofs admit of ready extension to the system (15).

On the other hand we may pass over from *difference* to *differential* equations by allowing the integer n to become infinite, the points x_0 and x_n however remaining fixed. In this way the various functions of the integral argument i become functions of a continuous argument x. We thus pass from the massless string weighted by n distinct particles to the string whose mass is continuously but unequally distributed throughout its extent, and from the radiation of heat in a row of vases to the conduction of heat in a heterogeneous bar. It is this latter problem which forms the subject of Sturm's second great memoir of 1836, while the extension of the results concerning the difference equation (15) to the differential equation

$$\frac{d}{dx}\left(K(x)\frac{dy}{dx}\right) - G(x)y = 0$$

is the subject of the first memoir of that year. It is worthy of notice that in both cases Sturm used the method of passing by a limiting process from a difference to a differential equation merely as a heuristic one, making indeed hardly a mention of it in the final memoirs, and treating the differential equations *directly* by methods which are the immediate generalizations of those he had used for the difference equations in his unpublished work.* A careful examination of the abstracts published in 1829 in the *Journal de Férussac* will show that all this, at least so far as it relates to the heat problem, was in his mind even at that early date.† It was probably developed to some extent in the unpublished paper of October 19.

It is not my purpose to discuss here the two great memoirs of 1836, although the richness of their detail tempts one to linger over many points which have, it is to be feared, rather escaped the notice of mathematicians. This richness of detail probably reflects a similar quality in the earlier unpublished

* Cf. Fredholm's derivation of the theory of integral equations as a limiting case from the theory of a system of linear algebraic equations. Like Sturm, Fredholm used this limiting process merely as a heuristic one for deriving both the results and the methods for their proofs.

† There is no evidence to show whether the more general difference equation (15) was considered at this time or only at a slightly later date.

researches, and much of the detail there could be readily reproduced. In these days, when new methods are being suggested for obtaining a few of the fundamental results of Sturm in their simplest forms, it is not out of place to remark that if one were to cut away from Sturm's memoirs everything except what is necessary to obtain *these* results, the few pages that would be left would in brevity, rigor, and directness easily stand comparison with anything which has so far been suggested to replace them.

Coupled with Sturm's name in all of this work on differential equations one often finds the name of his young friend Liouville. It is true that Liouville's work on these matters was hardly inferior in originality and power to that of Sturm himself; but it must be remembered that Sturm's work was practically complete, even to the writing of the two great memoirs, before Liouville's began, and that, except for alternative proofs which the latter gave for some of Sturm's results, and for a genial extension to certain differential equations of higher order, his work dealt with a single problem, of fundamental importance it is true, which had not been treated by Sturm,* namely the proof that the development of an arbitrary function which occurs in Sturm's papers is valid. We may therefore fairly speak of the Sturm-Liouville development according to normal functions, but these normal functions themselves, and almost everything relating to their theory,† are due to Sturm alone.

I have tried to show you how all of Sturm's most important work flowed naturally from his treatment of a single physical problem, not very important in itself perhaps, certainly of no great generality or largeness of scope. Sturm's genius showed itself first in his method of handling the problem where such purely formal skill as one associates with the names of Lagrange or Poisson is less in evidence than a constant intuitional visualization of the problem combined with a sense of accuracy uncommon in his day; secondly in his perception of the relation of this problem to other questions, and to the way in which he followed up his work into adjacent fields. The power of generalizing is not rare, as the huge bulk of our current mathe-

* One paper on this subject written in conjunction with Liouville is preserved to us in abstract (*Liouville's Journal*, vol. 2, p. 220) but was written after Liouville's first work on this subject.

† An exception should be made here of the asymptotic expressions for these functions for large values of the index. These important expressions are due to Liouville.

matical literature sadly reminds us; but one who like Sturm can seize on the important and simple modifications of a given problem has certainly one of the most essential elements of mathematical greatness.

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A SENSUOUS REPRESENTATION OF PATHS THAT LEAD FROM THE INSIDE TO THE OUTSIDE OF AN ORDINARY SPHERE IN POINT SPACE OF FOUR DIMENSIONS WITHOUT PENETRATING THE SURFACE OF THE SPHERE.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, April 28, 1911.)

THE logical or analytic existence of such paths—their existence in and for thought as distinguished from intuition or imagination—has been long familiar to every one, and may be made evident even to a freshman, so simple is the sufficient algebraic argument. But all efforts to envisage the paths are defeated completely.

It is the purpose of this note to show how the existence of the paths may be made evident to the intuition and even to the senses of sight and touch. The purpose is achieved by a simple transformation correlating the points of 4-space S_4 with the spheres of ordinary space S_3, including all spheres of real center and pure imaginary radius. In this way unintuitable situations in S_4, like that presented by the paths in question, are represented by intuitable analytic equivalents in S_3, and these equivalents may be rendered sensible by easily constructible physical models.

The simplest possible correlation of the kind in question is that in which the point (x, y, z, w) of S_4 and the sphere (of S_3) having (x, y, z) for center and $\sqrt{w}$ for radius shall be a pair of correspondents.

The representative in S_3 of a lineoid (an ordinary 3-space) $Ax + By + Cz + Dw + E = 0$ of S_4 is a linear complex of spheres such that, if (x_1, y_1, z_1, w_1) be a point of the lineoid, the

corresponding sphere of the complex is the sphere that has its center at (x_1, y_1, z_1) and its radius equal to $\surd w_1$. Those spheres of the complex that have a same radius, $r = \surd w$, have for locus of their centers the plane $Ax + By + Cz + Dr^2 + E = 0$; and the planes that thus correspond to different values of r are parallel.

In particular, the lineoid $w = k > 0$ which is parallel to the xyz-lineoid of reference, and on the positive side of it, is represented in S_3 by the sphere complex consisting of the real spheres whose center-locus is the totality of points of S_3 and all of which have the same radius $r = \surd k$: a complex naturally to be regarded as parallel to the sphere complex composed of the points of S_3 conceived as spheres of zero radius.

The line of S_4 determined by the points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) is represented in S_3 by a range of spheres determined by the two spheres that correspond respectively to the given points. The locus of the centers of these spheres is the line determined by the points (x_1, y_1, z_1), (x_2, y_2, z_2). A sphere of the range whose center is (x, y, z), where

$$x = (x_1 + \lambda x_2)/(1 + \lambda), \quad y = (y_1 + \lambda y_2)/(1 + \lambda),$$

$$z = (z_1 + \lambda z_2)/(1 + \lambda),$$

has for radius $r = \surd w$, where $w = (w_1 + \lambda w_2)/(1 + \lambda)$; so that in general the size of a sphere of a range varies as its center moves along the locus of centers of the range, i. e., as the sphere moves along the range, generating the range as the path of a moving sphere just as in S_4 the line corresponding to the range in S_3 is the path of a moving point.

The 4-dimensional sphere $x^2 + y^2 + z^2 + w^2 - R^2 = 0$ of S_4 is represented in S_3 by a certain quadratic complex of spheres, namely, that whose spheres have for locus of centers the ensemble of points within or on the sphere $x^2 + y^2 + z^2 - R^2 = 0$ and whose radii as the distance of the center from the origin increases vary for w positive from $\surd R$ to zero and for w negative from $i\surd R$ to zero. The spheres of the complex fall into two equal classes according as the radii are real or pure imaginary.

The pair of equations

$$(1) \qquad \begin{aligned} &x^2 + y^2 + z^2 + w^2 - R^2 = 0, \\ &w = k \quad (0 < k < R), \end{aligned}$$

determine an ordinary sphere (of S_4) that is immersed in the lineoid $w = k$. This sphere is represented in S_3 by a quadratic congruence of spheres, which may be called a sphered sphere. It is composed of real (visualizable) spheres whose centers have for their locus the sphere $x^2 + y^2 + z^2 = R^2 - k^2$ and whose radii are $\surd k$. This sphered sphere is immersed in the sphere complex corresponding, as above seen, to the lineoid $w = k$.

A point (x_1, y_1, z_1, w_1) of S_4 is inside, on, or outside of the sphere (1) according as $x_1^2 + y_1^2 + z_1^2 <, =,$ or $> R^2 - w_1^2$, provided $w_1 = k$. A point whose $w \neq k$ is properly neither inside, nor on nor outside the sphere (just as in ordinary space a point external to a given plane is properly neither inside, nor on, nor outside of a given circle of the plane). According as a point of S_4 is inside, on, or outside of the sphere (1), or has none of the three relations to it, so in S_3 the sphere that represents the given point is respectively inside, on, or outside the sphered sphere representing the sphere (1), or has none of the three relations to the sphered sphere.

Let (x_1, y_1, z_1, k) and (x_2, y_2, z_2, k) be two points respectively inside and outside of the sphere (1). The condition that a point

$$x = (x_1 + \lambda x_2)/(1 + \lambda), \quad y = (y_1 + \lambda y_2)/(1 + \lambda),$$

$$z = (z_1 + \lambda z_2)/(1 + \lambda), \quad w = k,$$

of the line determined by the given points shall be on the sphere (1) has a positive discriminant, showing that the line pierces the sphere in two real points. This case is the familiar one that occurs in the point geometry of S_3 and thus lies in the field of geometric intuition. In the sphere geometry of S_3 the case is represented by a sphere range determined by two spheres having $\surd k$ for radii and situated respectively inside and outside of the sphered sphere corresponding to sphere (1). It is essential to note that all the spheres of the range are of the same size and that the size is that of the spheres composing the sphered sphere, so that, as a sphere traverses the range, it twice coincides with a sphere of the sphered sphere; that is, the range as a path of a moving sphere twice penetrates the (surface of the) sphered sphere.

Now consider the lines of S_4 determined respectively by the point pairs (x_1, y_1, z_1, k), (x_3, y_3, z_3, w_3), and (x_3, y_3, z_3, w_3), (x_2, y_2, z_2, k), where $w_3 >$ or $< k$, say $w_3 > k$. No point of the

former line is on sphere (1), for such a coincidence of points requires that $(k + \lambda w_3)/(1 + \lambda) = k$, a condition that is satisfied only when $\lambda = 0$, which yields the point (x_1, y_1, z_1, k), but this point is by hypothesis not on but inside sphere (1). For a like reason no point of the latter line is on sphere (1). Hence the broken line running from the first to the second and from this to the third of the given points is a path (for points) leading from the inside to the outside of sphere (1) without penetrating the surface. It is such analytically evident but unintuitable paths whose existence is to be made evident to spatial intuition and to sight and touch. The means thereto is now obvious. In S_3 the two spheres representing the first and last of the given points are respectively inside and outside of the sphered sphere representing sphere (1). The sphere representing the point (x_2, y_2, z_2, w_2) is neither inside nor on nor outside of the sphered sphere, a fact intuitively manifest because the sphere in question is larger than the spheres $(r = \sqrt{k})$ composing the sphere complex in which the sphered sphere is immersed. As to the broken sphere range representing the broken line, it is intuitively evident that it does not penetrate the sphered sphere (whose spheres are all of radius $\sqrt{k}$) for the spheres composing the first part of the broken range increase in size from that of the initial sphere (of radius $\sqrt{k}$ and lying inside of the sphered sphere) up to the final sphere (radius $\sqrt{w_3}$) of that part, whilst the spheres of the second part decrease in size from that of the sphere just mentioned down to the final sphere of the range, this sphere being outside of the sphered sphere and having $\sqrt{k}$ for radius.

To construct a model rendering the matter sensibly evident it is plainly sufficient (1) to distribute throughout a small region of space a small number of small spheres, fixing them in position as by slender vertical steel rods, some of the spheres being of equal radius r, one or more of greater radius and one or more of less, some of the spheres intersecting (though this is not indispensable) and two or more of them, if transparent, being concentric (though this feature may be omitted); (2) to construct a spherical frame (say of wire) that shall include one or more of the spheres of radius r and exclude some of them, and to locate a number of spheres of radius r so that their centers shall be points of the frame and be suitably distributed upon it; (3) to join by a rod the centers of two spheres of radius r, one of them inside the frame, the other outside, and to place

a number of spheres of radius r so that their centers shall be on the rod and be suitably distributed on it; (4) finally to join by a rod the center of a sphere of radius r inside the frame with the center of a sphere of radius $> r$ (it might as well be $< r$) and by a rod the center of the latter sphere with a sphere of radius r outside the frame, and, then, to locate a number of spheres so that the centers of some of them shall be on one of the rods and those of the rest on the other, the spheres on the former rod suitably increasing in size as their centers approach that of the sphere common to the two rods, whilst the spheres on the second rod suitably decrease in size as their centers recede from the common point of the rods.

The model obviously enables one to distinguish by sight or by touch a sphere range representing a line of S_4 that joins a point inside of an ordinary sphere to an outside point without penetrating the surface from a sphere range representing the case where penetration occurs.

COLUMBIA UNIVERSITY,
April, 1911.

A DIRECT PROOF OF THE THEOREM THAT THE NUMBER OF TERMS IN THE EXPANSION OF AN INFINITE DETERMINANT IS OF THE SAME POTENCY AS THE CONTINUUM.

BY DR. N. J. LENNES.

(Read before the American Mathematical Society, February 25, 1911.)

THE following rather obvious theorem is thought of sufficient interest to warrant the publication of a direct proof.

THEOREM: *The number of terms in an infinite determinant is of the potency c of the continuum.*

Proof. It is well known that every infinite determinant can be brought into the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \end{vmatrix} \quad \text{or } |a_{ij}|.$$

Let the elements of the determinant be ordered according to the sum of their subscripts. In case this sum is the same for two elements these are ordered according to the value of i. Then we get an infinite sequence of elements a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, ... In each infinite product of the expanded determinant order the elements which form its factors according to the same principle.

Divide a segment AB into two equal segments AC and CB, CB in two equal segments CD and DB, and so on by sequential halving of the last segment obtaining an infinite sequence of segments whose end points approach B as a limit. Repeat this construction on each of the segments AC, CD, ..., also on each of the segments thus obtained and so on indefinitely. Consider now a term $a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \ldots$ of the expanded determinant. Let $a_{i_1 j_1}$ be the k_1th element of the determinant when these are ordered as above. Now there is a perfectly definite number of elements of the determinant between $a_{i_1 j_1}$ and $a_{i_2 j_2}$ which may occur with $a_{i_1 j_1}$ as factors in a term of the expanded determinant, that is, which are not in the same row or column with $a_{i_1 j_1}$. Let this number be $k_2 - 1$. There is further a definite number of elements between $a_{i_2 j_2}$ and $a_{i_3 j_3}$ which may occur in a product with $a_{i_1 j_1}$, $a_{i_2 j_2}$. Let this number be $k_3 - 1$ and so on.

We now assign the term $a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \ldots$ to a sequence of segments on AB in the following manner: The first segment of the sequence is the k_1th segment of the sequence AC, CD, ... The second segment is the k_2th segment of the division of the segment just selected, the third is the k_3th segment of the division of the segment last selected, and so on indefinitely. We thus obtain a sequence of segments corresponding to the term $a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \ldots$, such that every segment of the sequence lies on all that precede it and also such that the lengths of the segments become indefinitely small as we proceed. Hence this sequence determines a definite point which is a common end point of all the segments of the sequence beyond a certain one or else lies within every segment of the sequence. Note that if it is a common end point of such segments then it is a left end point. In this manner we obtain on AB a point corresponding to every term of the expanded determinant. It remains to show: (1) that we thus obtain a term corresponding to every point on AB, (2) that two points thus set in correspondence to different terms of the expanded determinant are not identical.

(1) Let Q be any point on AB. In the sequence of segments

AC, CD, . . . , there is one segment within which Q lies or of which it is the left end point. (We conceive of the segment AB as extending from left to right.) Further let Q be the left end point of or lie within the k_2'th segment of the division of the k_1'th segment of AB, and so on. Thus we obtain a sequence of numbers k_1', k_2', k_3', . . . , corresponding to a definite sequence of segments which determine the point Q. The term $a_{i_1'j_1'}a_{i_2'i_2'}a_{i_3'i_3'}$ . . . is set in correspondence with the point Q by the process described above, provided $a_{i_1'j_1'}$ is the k_1'th element of the determinant, $a_{i_2'j_2'}$ the k_2'th element after $a_{i_1'j_1'}$ of those which can occur with it in a term of the expanded determinant, and so on.

(2) Two different terms $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}$ . . . and $a_{i_1'j_1'}a_{i_2'j_2'}$ $a_{i_3'j_3'}$ . . . of the expanded determinant cannot be set in correspondence to the same point by this process. Suppose their kth factors are different. For the sake of simplicity of statement we suppose the first factors $a_{i_1j_1}$ and $a_{i_1'j_1'}$ are different. Then the corresponding points P and P' are end points of or lie within different segments of the sequence AC, CD, . . . , obtained by the division of AB. Hence these points can be identical only in case one, as P, is a right end point of one of these segments and P' the left end point of the next segment. But we noted above that in case a point determined by one of these sequences is a common end point of the segments of the sequence determining it, then it must be a left end point of such segments. Hence P and P' cannot be identical.

COLUMBIA UNIVERSITY,
January 3, 1911.

SHORTER NOTICES.

The Theory of Sets of Points. By W. H. YOUNG and GRACE CHISHOLM YOUNG. Cambridge, University Press, 1906. xii + 316 pp.

THIS volume consists of a systematic presentation of the theory of sets of points. The titles of the chapters are: Rational and irrational numbers, Representation of numbers on the straight line, The descriptive theory of linear sets of points, Potency and the general idea of a cardinal number, Content, Order, Cantor's numbers, Preliminary notions of plane sets,

Regions and sets of regions, Curves, Potency of plane sets, Plane content and area, Length and linear content.

While the main body of the subject matter is thus seen to consist of those well-known elements of point set theory developed by Cantor and others, there is nevertheless considerable material which is original with the authors. As stated in the preface: "The writing of the book has necessarily involved attempts to extend the boundaries of existing knowledge and to fill in gaps which broke the connection between isolated parts of the subject."

Chapter I on rational and irrational numbers contains only eight pages and consequently a good deal is here necessarily left to the reader. The chapter ends with the theorem of Liouville.

In Chapter II a correspondence preserving order is set up by means of projection between the numbers of the real number system and the points of a line. This enables the author, without employing the notion of measurement, to use interchangeably the points on the line and the real numbers in discussions involving order and the number of points (potency) of a set. This correspondence cannot however furnish a basis for the treatment of content of sets of points since the relative lengths of segments on a line would depend upon the choice of the points 0, 1, and ∞. That is, it would depend upon the point at which one starts measuring and also upon the unit of measure. Clearly the property of relative lengths of segments on a line must be a property of those segments themselves and cannot depend on any arbitrary choice. In Chapter V, where content is considered, no mention is made of this fact and the unwary reader might be led to believe that this projective correspondence is all that is needed. Theorems relating to order, limiting points, character of sets as to density and potency are all valid without a basis in stronger congruence assumptions but, in general, theorems comparing the lengths of segments are not.

Chapter III deals mainly with what are called derived and deduced sets. A derived set is the set consisting of all the limit points of a set. If A_1 is a derived set of A and A_2 a derived set of A_1 and so on indefinitely then the set common to all the sets $A_1, A_2, \cdots$ is the deduced set of this set of sets of points. In general the process of taking all points common to an infinite series of sets of points is called deduction. An example is given of sets which have an infinite series of derived sets, all different, a first deduced set which in turn has an infinite series

of derived sets all different, this series a deduced set and so on indefinitely. This serves as a concrete example in which the transfinite numbers of Cantor appear naturally and is later (Chapter VIII) referred to in connection with the formal treatment of these numbers. This is one instance of what seems one of the noteworthy features of this book, viz., the large number of simple and highly instructive examples.

Chapter IV treats of potency and the generalized idea of cardinal numbers. Several theorems due to Mr. W. H. Young are here given. Among these are the theorems on overlapping intervals covering any set and the generalized Heine-Borel theorem.*

We note the following definitions: "If $G_1, G_2, \cdots$ be a series of sets of points such that, for all values of n, G_n contains G_{n+1} and if G be the set of all the points common to the sets G_n, G is called the *inner limiting* set of the series.$\cdots$" "If in the preceding G_n is contained in G_{n+1} and G be the set such that every set G_n is contained in G, while every point of G belongs to some definite G_n, G is said to be the *outer limiting* set of the series." Surely this terminology is much to be preferred to the less natural expressions greatest common divisor and lowest common multiple used by Cantor and his continental followers. The most important innovation in this chapter is the treatment of derived and deduced sets (adherences and coherences) without the use of Cantor's numbers. The doubt expressed in the *Fortschritte der Mathematik*, page 530, volume 34, as to the possibility of treating this subject without the use of Cantor's numbers cannot possibly have any foundation. Explicitly the question is as to whether a theory which is a special case of a more general theory may not be discussed on its own merits without reference to that more general theory. Stated in this form the question is trivial. In the case of a difficult subject like the transfinite numbers in their full generality it would appear good procedure to study a less general case of the subject first. In so far as this more special case involves properties that are taken account of in the case of the more general treatment of transfinite numbers, in so far the discussion must involve notions contained in the more general treatment. The remark by Schoenflies "H. W. Young benützt bei seinen Beweisen, die sachlich darauf hinauslaufen, eine wirkliche sukzessive

* The latter was also given by H. Lebesgue in his thesis, Paris, 1902, the same year that Young published the theorem in the *Proc. L. M. S.*

Analyse der Punktmengen vorzunehmen, nicht ausdrücklich aber doch stillschweigend die Cantorschen Zahlen der zweiten Klasse ···"* is therefore also trivial.

The definitions of content, Chapter V, are novel. "The content I_p of a closed set of points is the difference between the content of the fundamental segment and that of those intervals of the fundamental segment which contain no points of the set." (It is proved earlier that every closed set may be obtained as the remainder of a segment when a certain set of open segments, possibly an infinite set, has been removed.) The *inner content* of any set is defined as the upper limit of the contents of its closed components. The *outer content* is the lower limit of the content of all sets of intervals covering the set. The outer content or measure of Young is identical with that of Lebesgue, while the inner measure of Lebesgue is obtained by subtracting the exterior measure of the complementary set from the whole segment. It is rather difficult to decide which of these two definitions serves the purpose in hand more directly. Schoenflies with his usual air of finality remarks: "Statt Lebesgues natürlicher Definitionen stellt H. W. Young die folgenden künstlichen an die Spitze."† It is then shown that the question as to whether or not all sets are measurable (have identical inner and outer content) depends upon whether or not there exist two open sets with no point in common, each of inner content zero, whose sum is a closed set of content not zero.

The following theorem due to W. H. Young is worthy of note: "If $G_1, G_2, \cdots, G_n, \cdots$ is a sequence of sets of points, each of which sets is a component of a closed set of finite content l, and if the interior measure of each of the sets $G_1, G_2, \cdots, G_n, \cdots$ is greater than a fixed number e, then there exists a set of points of interior measure $\geqq e$, and of the power of the continuum, such that each point of the set belongs to an infinite number of the given sets." This is regarded by the authors as one of the most important in the whole theory of linear content.

The subject of content is completed in Chapter XII, where the content and area of plane regions are discussed in full detail and the way is pointed to a similar discussion for the n-dimensional region.

Chapters VI and VII are devoted to order and the Cantor numbers respectively. Particular care is taken to define order

* Bericht über Punktmannigfaltigkeiten, zweiter Teil, page 76.
† Loc. cit., p. 88.

with respect to the set of points whose order is being defined and not with respect to the fundamental set, e. g., the continuum.

In Chapter VIII a brief resumé is given of the kinds of (1, 1)-correspondence possible between the points of a planar and a linear continuum, "space filling" curves, etc. The following statement is evidently meant to indicate the authors' point of view in regard to the "crinkly" curves but does not seem very definite: "··· the concept of a curve, as such, regarded as a set of points, must surely be recognized as a conglomeration of ideas of which that of order is only one, and is from many points of view a subservient one. A curve like any other plane set of points, has a *form*, (the italics are the reviewer's) and it is in many respects this form which is its most interesting characteristic" (page 168). This might conceivably be a very interesting statement if we knew what definite concept to attach to the word "form."

The notion of *region* is developed from the simplest region, viz., the triangular region. A region is namely a part of a plane which may be tiled over by a set of triangles (finite or infinite). A region may or may not contain rim points (points which are limit points of points not in the region) while a *domain* is definitely defined to be a region which does not contain a single rim point. This definition of domain is the basis for one of the several inaccurate statements which to a certain extent mar this chapter.

The statement "The part of a domain left over after removing a domain contained in the first (difference of two domains) is a domain or domains" (page 196) is obviously not true. Again Theorem 10, page 193, should read: "If the regions have *at least* two common points, and also any further common points lie on the same straight line as those two, then (1) the span in one direction (viz., that perpendicular to the line of common points) decreases without limit; (2) the spans in any other direction have a positive lower bound."

The corollary under Theorem 5, page 187, is not true. Besides the set specified in the corollary there may be a set of isolated points or a discrete set with limit points. Page 187, line 26, read "spans" for "span" and "limit" for "limits." In the definition of disc, page 188, insert after "meets it" the words "in an internal point." The definition of connected set leads to curious results: "A set of points such that, describ-

ing a region in any manner round each point and *each limiting point* (the italics are the reviewer's) of the set as internal point, these regions always generate a single region, is said to be a connected set, provided it contains more than one point. Hence if a set is connected, the set got by closing it is connected and vice versa" (page 204). This definition, which is shown to be equivalent to that given by Cantor in purely metric terms, makes the inside of triangle connected with its outside, and in general two sets neither of which contains a limit point of the other may form one connected set.

The following statement requires lenient construction to make it true: "Given any set of regions, a (countable) set of non-overlapping regions is uniquely determined, having the same internal points as the given set" (page 199). What meaning can be given to the words "uniquely determined" to make this hold?

These various errors are apparently due to temporary inattention rather than to lack of insight. In no case do they lead to grave results.

A curve is defined as follows: "A plane set of points, dense nowhere in the plane, such that, given any norm e, and describing around each point of the set a region of span less than e, these regions generate a single region R_e, whose span does not decrease indefinitely with e, is called a curved arc, or shortly a curve" (page 219). This definition is essentially new, differing radically from that of Schoenflies who regards a curve as the frontier of a region, or from that of Veblen which is given in terms of order and linear continuity.* As will readily be noticed, this curve is very different from an arc of a Jordan curve in that it may consist of a network of such curves as complicated as we please, —indeed of infinite number of "branches" of such curves. In dealing with the separation of the plane by polygons and curves a great deal is left to the reader. The general outline of a fuller treatment may be said to be given but it would be no mean task to fill it out in detail.

Chapters XII and XIII, dealing with plane content and area, and length and linear content, make such use of the chapters that precede as is logically necessary, and hence the character of these last chapters of the book is practically determined by the earlier parts. Chapters IX, X, and XI are not required for the reading of these last chapters and that on plane content.

* *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 83–98.

The figures employed are all of an elementary nature, circles, rectangles, triangles, and squares. On the other hand some of the results of these chapters are stated in the book for the first time. Schoenflies (Bericht, II, page 93) has incorrectly reported the decisive examples on pages 275–283. The surprising result that a countable closed set of points in the plane may have positive linear content, which may even be infinite, has entirely escaped comment in the "Bericht." An appendix deals with various questions that arose during the printing of the book but too late for insertion in the main body of it. At the end a very full bibliography is given,—so full as to be rather discouraging. It would be of real service if the book contained in a short compass a statement of the contributions to point set theory by the different investigators.

Throughout there are many well constructed figures which assist the reader very materially. The large number of problems exhibiting a multitude of phases of the subject bear eloquent witness to the care with which the authors themselves have mastered the subject and the great amount of energy expended in writing the book.

Inasmuch as this is practically the only treatise of its kind (Schoenflies's Bericht apparently having a quite different purpose), it is difficult to judge how greatly it differs from the riper treatises which are bound to come in the future. But we are surely justified in saying that the authors have done the cause of mathematics a real service by placing at the disposal of the student a treatment of point sets exceptionally readable and with unimportant exceptions entirely trustworthy.

A very considerable number of misprints have been detected.

N. J. LENNES.

Leçons élémentaires sur la Théorie des Fonctions analytiques. 2nd edition. By ÉDOUARD A. FOUËT. Vol. I: *Les Fonctions en général.* 1907. xiii + 112 pp. Vol. II: *Les Fonctions algébriques. Les Séries simples et multiples. — Les Intégrales.* Paris, Gauthier-Villars, 1910. xi + 263 pp.

A COMPARISON of the first and second editions shows that the author has completely revised his work, adding new material to include some of the latest developments, enlarging some of the subjects already treated and rewriting other portions. The one volume of the first edition has been expanded into two.

In the first volume the author seeks to give a general intro-

duction to analytic functions, laying special emphasis upon the fundamental ideas, and indicating the growth of these ideas from the simplest naive notions to the present extended conceptions. Thus, the growth of the idea of function is traced from the earliest analysts, who restricted the use of the word to the powers of a quantity, through the various stages represented by Euler, Cauchy, Riemann, Fourier, and Dirichlet, to the present general conception which is broad enough to include a correspondence which represents a space-filling curve.

After treating the subjects of number and function, the author takes up at some length the principles of assemblages. He then considers the principal types of functions and their classification. He introduces the subject of the general analytic function, gives the Cauchy-Riemann partial differential equations, studies conformal representation, considers various singularities, introduces groups and applies them to periodic and automorphic functions.

The second volume is a study of functions from the three points of view suggested by the names of Riemann, Weierstrass, and Cauchy. The first chapter, on algebraic functions, is dominated by the geometric point of view. It studies various transformations, the fundamental properties of algebraic functions, and contains an exposition of the Riemann surface.

The subject of series is treated very fully. A general discussion of convergence is followed by a treatment of the power series, infinite products, trigonometric series, and divergent series (presenting largely the points of view of Poincaré and Borel). The theory is followed by applications to e^z, $\sin z$, $\zeta(z)$ of Riemann, etc., and by a consideration of the hypergeometric series.

Series in two or more variables are also studied, and applied to various transcendental functions, e. g., the ζ, γ, γ' functions of Weierstrass, the θ functions of Jacobi, etc.

The last chapter deals with functions as defined by the definite integral, the epoch-making conception of Cauchy, which, according to Hermite, is the most fruitful conception in mathematics. The objective of this chapter is the integrals of Cauchy and Taylor's series.

Throughout the two volumes we find a double purpose: the author seeks to give an elementary account of the subject from the most modern standpoint; and aims, at the same time, at completeness. He accomplishes his purpose very well by the

introduction of copious footnotes which deal with sources, historical information, difficult demonstrations, interesting generalizations, and conceptions too advanced for the text proper. Thus, the discussion of the theory of assemblages is supplemented in the footnotes by reference to transfinite assemblages; and the ordinary presentation of the integral is supplemented by the introduction of upper and lower integrals.

The comprehensive, suggestive, critical footnotes greatly enhance the value of the work.

The volumes under review are especially valuable for those who wish a thorough treatment of the fundamental conceptions, and an introduction to the latest ideas, for the author has not only given a sketch of these phases of the subject, but has also indicated many sources.

GEO. N. BAUER.

Lectures de Mécanique: La mécanique enseignée par les Auteurs originaux. Par E. JOUGUET. Deuxième partie: *L'Organisation de la Mécanique.* Paris, Gauthier-Villars, 1909. 284 pp.

MANY of the mathematicians who teach the elements of mechanics may have more or less serious *arrières pensées* relative to the way in which they have presented the fundamental concepts to their students, and they may form many a good resolution as to the severe logical thinking they will expend upon the subject to the end that the next time they teach it the presentation may gain much in completeness and consistency. To all such Jouguet's Lectures are a godsend,—not that all the difficulties of the doubting ones will be relieved by the perusal of the work, but that the doubts and perplexities of the great creators of mechanics, and the way they settled them or at least thrust them aside, are here detailed. For the plan of the work, as the subtitle indicates, is to teach (the foundations of) mechanics by (large extracts from) the original authors. We may say that Jouguet selects his quotations well and makes each one sufficiently long to be intelligible of itself; but one must add that his own careful critical comments are very helpful toward the fullest interpretation both of the material cited and of the subject itself.

The work consists of three parts: the first, which is not at hand, called La naissance de la mécanique; the second, which is under review, entitled L'organisation de la mécanique; the

third, which as yet is only projected, to be designated Les bornes de la mécanique. Although the first and last parts must of necessity throw much light upon the second, it should be observed that this second, which has chiefly to do with the dynamics of a particle and rigid body and which as such bears most closely on the problems of interest to the teacher, may without difficulty be read alone. And the reading is highly to be recommended. The extremely logical may long for a specific set of postulates by virtue of which the subject matter and terms of elementary mechanics shall be defined; but though the actual establishment of such a set be a worthy scientific objective point, one should recognize that it may have much of the sterility of any puristic accomplishment,—the instructive thing is to contemplate the foundations of a subject from various angles, and this is precisely what Jouguet makes easy for us in reference to mechanics.

We have written from the point of view of the teacher, but the student, the physicist, the philosopher will find much of interest to them in these pages, and the logician who would construct a set of postulates for mechanics which should be of real value to teacher and pupil as well as of scientific exactness cannot safely neglect the points of approach which the race, through its geniuses, has shown to be the genius of the race.

E. B. WILSON.

CORRECTION.

PROFESSOR D. R. Curtiss calls our attention to the following erratum in his paper published in the BULLETIN in June last:

Page 466. In the formula for the determinant $\Delta_n(x, \xi)$, the index $n-2$ should, throughout the last row, be replaced by $n-3$.

NOTES.

THE concluding (July) number of volume 12 of the *Annals of Mathematics* contains the following papers: "The harmonics of a stretched string vibrating in a resisting medium," by C. R. DINES; "The mixing effect of surface waves," by C. S. SLICHTER; "Characteristics of two partial differential equations of order one," by C. A. NOBLE; "The differential equation of the third order with a quadratic equation between the integrals," by S. EPSTEEN; "Relations among some cyclotomic numbers," by T. HAYASHI.

THE eighty-first annual meeting of the British association for the advancement of science was held at Portsmouth, England, August 30 to September 6, under the presidency of Sir William Ramsey. Professor H. H. TURNER was chairman of section A, mathematics and physics.

THE fortieth annual meeting of the French association was held at Dijon, July 31 to August 5, with Inspector Ch. Lallemand as president. E. BELOT was chairman of the mathematical section.

THE fifth annual meeting of the Italian association will be held at Rome, October 12 to 18; Professor G. Ciamician is president, and Professor G. CASTELNUOVO chairman of section A.

THE second congress of Scandinavian mathematicians was held at Copenhagen, August 28–31.

THE organizing committee of the Fifth international congress of mathematicians, to be held at Cambridge, England, August 22–28, 1912, has issued a circular containing the names of the 46 members of the international committee and giving the following items of information:

"Special arrangements will be made for the consideration of the reports of the international committee appointed at the congress in Rome to enquire into the teaching of mathematics.

The committee has commenced to organize a series of lectures which shall give some idea of the present state and progress of the principal branches of mathematics, including its applications. The speakers already chosen are E. BOREL, E. W. BROWN, A. KNESER, E. G. H. LANDAU, Sir J. LARMOR, and Sir W. WHITE.

Their subjects will be announced later. The congress will be divided into four sections: I, arithmetic, algebra, analysis; II, geometry; III, applications; IV, philosophical questions."

AMONG the prizes awarded by the Paris academy of sciences at the close of the second semester, the following were for achievements in pure and applied mathematics: The Lalande prize (fr. 540) to Dr. Lewis Boss, of the Albany Observatory; the Pontécoulant prize (fr. 700) to Dr. L. SCHULOF, for his theory of comets and the completion of the lunar tables; the Francoeur prize (fr. 1000) to E. LEMOINE for his work in geometry; the Poncelet prize (fr. 2000) to M. RATEAU for progress in mechanics; the Montyon prize (fr. 700) to M. JOUGUET, for his work in thermodynamics. A general prize of fr. 10,000 was awarded to the estate of the late Professor J. TANNERY for his work on the theory of functions and particularly for his services as secretary to the Ecole Normale.

THE section of physics and mathematics of the royal society of Naples announces the following prize problem:

"A new contribution to the theory of differential forms of general order and degree."

Competing memoirs should be written in Italian, French or Latin, and sent to the secretary of the academy before June 30, 1912. The prize is 500 lire.

FROM the Dr. Elsa Neumann foundation the University of Berlin will award a prize of 1000 marks to the author of the most meritorious contribution in mathematics or physics submitted to the philosophical faculty of the university during the year 1911. The award will be made February 18, 1912.

THE Provisional Report of the national committee of fifteen on a geometry syllabus, has now been issued in pamphlet form. It is a reprint of the partial reports published in *School Science and Mathematics*, during April, May, and June, 1911, and was submitted in its present form to the National educational association at its San Francisco meeting, July 8–14, 1911. The report contains a historical introduction of 25 pages prepared by Professor F. Cajori; a section on logical considerations including axioms, definitions, treatment of limits, time and place in the curriculum, purpose of the study of geometry; a section on the grading and distribution of exercises; a section on types

of courses for special classes of students and on preliminary inductive courses in the grades; and finally the syllabus itself, exhibiting by means of different forms of type the varying degrees of emphasis which may properly be attached to the various theorems.

AT the meeting of the International commission on the teaching of mathematics held at Milan, Italy, September 18–21, 1911, the central committee received reports of the various national sub-committees and formulated plans regarding the submission of the final report to the Fifth international congress of mathematicians to be held at Cambridge, England, next August. From the reports of the various sub-committees the following questions were discussed at length:

A. To what extent can the systematic presentation of mathematics in the advanced schools (colleges, lycées, Gymnasien, Realschulen, · · ·) be taken account of? Is it wise to emphasize the fusion of different branches in such instruction?

B. What is the proper instruction in mathematics both pure and practical, for students of physics and the natural sciences?

THE annual list of American doctorates published in *Science* presents for the academic year 1910–1911, 437 names, of which 239 are credited to the sciences. The following 26 successful candidates offered mathematics as major subject (the titles of the theses are appended): H. L. AGARD, Yale, "The extension of some theorems in the theory of sets of points in n-dimensional space"; T. B. ASHCRAFT, Johns Hopkins, "Quadratic involutions on the plane rational quartic"; Miss C. L. BACON, Johns Hopkins, "The Cartesian oval and the elliptic functions"; R. P. BAKER, Chicago, "The problem of the angle bisectors"; Miss I. BARNEY, Yale, "Line and surface integrals"; W. H. BATES, Chicago, "An application of symbolic methods to the treatment of mean curvature in hyperspace"; F. W. BEAL, Princeton, "Associated normal congruences"; Miss A. D. BIDDLE, California, "Constructive theory of the unicursal plane quartic by synthetic methods"; P. P. BOYD, Cornell, "On the perspective Jonquières involutions associated with the (2, 1) ternary correspondence"; D. BUCHANAN, Chicago, "A class of periodic solutions of the problem of three bodies, two of equal mass, the third moving on a straight line"; B. H. CAMP, Yale, "The convergence of singular integrals"; R. D. CARMICHAEL, Princeton, "Linear difference equations and their analytic solu-

tions"; J. L. JONES, Yale, "Number concept"; S. LEFSCHETZ, Clark, "On the existence of loci with given singularities"; L. LINDSAY, Syracuse, "The minors of a compound determinant"; W. R. MARRIOTT, Pennsylvania, "The determination of the order of the groups of isomorphisms of the groups of order p^4, where p is a prime"; W. O. MENDENHALL, Michigan, "On the characteristic properties of sum formulas in the theory of divergent series"; W. J. MONTGOMERY, Clark, "Singularities of twisted quintic curves"; L. O'SHAUGHNESSY, Pennsylvania, "The integrability of the differential equation representing the sum of a family of series"; A. D. PITCHER, Chicago, "The interrelations of eight fundamental properties of classes of functions"; H. W. REDDICK, Columbia, "Systems of tautochrones in a general field of force"; E. B. STOUFFER, Illinois, "Invariants of linear differential equations with applications to projective differential geometry"; S. E. URNER, Harvard, "Certain singularities of point transformations in space of three dimensions"; Miss W. P. WEBBER, Cincinnati, "On the construction of doubly periodic functions which have singular points (polar and essential) in the period parallelogram"; Miss M. B. WHITE, Chicago, "The dependence of the focal point on curvature in space problems of the calculus of variations"; W. A. WILSON, Yale, "Theory of point aggregates applied to Lebesgue integrals."

HARVARD UNIVERSITY.—During the academic year 1911–12 Professor G. A. Bliss of the University of Chicago will spend the months October to December, and Professor Max Mason of the University of Wisconsin the months February to June as Lecturers in Mathematics at Harvard. Besides taking part in more elementary instruction, they will give the following advanced courses: Professor BLISS: Differential geometry of curves and surfaces, three hours; Partial differential equations, three hours. Professor MASON: Dynamics of rigid and elastic bodies, three hours; The electron and the electro-magnetic field, three hours.

The following advanced courses are also announced:

By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra, three hours; Trigonometric series, introduction to spherical harmonics, the potential function, three hours with Professor B. O. PEIRCE.—By Professor B. O. PEIRCE: Hydromechanics, two hours, first half-year.—By Professor W. F. OSGOOD: Advanced calculus, three hours; Theory

of functions, three hours.—By Professor M. BÔCHER: Vector analysis, three hours, first half-year; Finite differences and difference equations, three hours, second half-year.—By Professor C. L. BOUTON: The elementary theory of differential equations, three hours, first half-year; Differential equations and Lie's theory, three hours.—By Professor J. L. COOLIDGE: Probability, three hours; Line geometry, three hours.—By Dr. D. JACKSON: Infinite series and products, three hours, first half-year; Advanced algebra, three hours, second half-year; The theory of numbers including the theory of ideals, three hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1911–1912. Courses in algebra, analytic geometry, projective and descriptive geometry, and elementary courses in the calculus, mechanics, astronomy, and geodesy are not included:

UNIVERSITY OF BOLOGNA: By Professor C. ARZELÀ: Higher mathematics, three hours.—By Professor P. BURGATTI: Dynamics of rigid bodies with application to planetary motion, equilibrium of a rotating fluid mass, three hours.—By Professor L. DONATI: Account of the different electromagnetic theories and of the principle of relativity, three hours.—By Professor S. PINCHERLE: Theory of analytic functions, linear differential equations, three hours.—By Professor U. SCARPIS: Operation groups and their application to the theory of numbers, three hours.

UNIVERSITY OF CATANIA.—By Professor M. DE FRANCHIS: Geometry on algebraic curves and surfaces, hyperelliptic surfaces, four hours.—By Professor G. PENNACCHIETTI: Dynamics of rigid bodies, mechanics of deformable media, four hours.—By Professor C. SEVERINI: Integral equations and their applications to analysis, four hours.—By ———: Mathematical physics, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Elementary theory of functions of one and more complex variables, problem of uniformization of polydromic functions, four hours.—By Professor G. LORIA: Algebraic and transcendental curves and surfaces, three hours.—By Professor O. TEDONE: Integration methods of Riemann-Volterra, application to boundary problems, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics during the middle ages (XIII to XVI century), three hours.—By Professor A. DEL RE: Grassmann's analysis in n-dimensions with application to geometry and mechanics in spaces of constant curvature, four and one-half hours.—By Professor R. MARCOLONGO: Application of vector-homographics to hydrodynamics, three hours.—By Professor D. MONTESANO: General theory of algebraic surfaces, surfaces of the third and fourth order, four and one-half hours.—By Professor E. PASCAL: Selected chapters of advanced analysis, three hours.—By Professor E. PINTO: Electrostatics, four and one-half hours.—By Professor G. TORELLI: Analytic theory of numbers (advanced part), three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: General theory of functions, elliptic functions, four hours.—By Professor U. CISOTTI: Mathematical theory of elasticity with technical applications, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Waves in their different meanings, four and one-half hours.—By Professor G. RICCI: Absolute differential calculus with applications, four hours.—By Professor F. SEVERI: Theory of algebraic functions of two variables and of their integrals (advanced part), four hours.—By Professor G. VERONESE: Foundations of geometry (advanced part), four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: General theory of analytic functions, algebraic functions of one variable, three hours.—By Professor M. GEBBIA: Elasticity, wave theory of light, four and one-half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one-half hours.—By Professor A. VENTURI: Figure of planets and especially of the Earth, Pratt's, Stoke's, Helmert's theories, shell-tides, gravity, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Birational transformations in the plane and in space, three hours.—By Professor F. GERBALDI: Functions of a complex variable, elliptic functions, three hours.—By Professor G. VIVANTI: Calculus of variations, integral equations, three hours.—By ———: Mathematical physics, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry

on an algebraic surface (advanced part), three hours.—By Professor L. BIANCHI: Arithmetic theory of quadratic (binary and ternary) forms, principles of analytic arithmetic, arithmetic of algebraic fields, four and one-half hours.—By Professor U. DINI: Spherical and Bessel's functions, four and one-half hours.—By Professor G. A. MAGGI: Equilibrium and motion of elastic bodies, application to optics, four and one-half hours.—By Professor P. PIZZETTI: Spherical astronomy, determination of planetary orbits, precession, nutation, theory of the tides, four and one-half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Elementary differential properties of curves and surfaces, three hours.—By Professor G. CASTELNUOVO: Differential geometry, three hours.—By Professor G. LAURICELLA: Boundary problems, three hours.—By Professor L. ORLANDO: Mathematical and physical grounds of aerial navigation, three hours.—By Professor L. SILBERSTEIN: Principles of thermodynamics, electromagnetics and optics, mechanics according to the principle of relativity, three hours.—By Professor V. VOLTERRA: Optics, three hours; application of mechanics to geophysical questions, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Equilibrium of a rotating fluid mass, three hours.—By Professor G. FUBINI: Theory of partial differential equations in both the real and the complex field, Cauchy's and boundary problems, three hours.—By Professor G. SANNIA: Geometrical applications of the theory of algebraic forms, one hour.—By Professor C. SEGRE: Continuous groups of transformations, three hours.—By Professor C. SOMIGLIANA: Propagation of heat, thermodynamics, three hours.

PROFESSOR L. FEJÉR, of the University of Klausenburg, has accepted a professorship of mathematics at the University of Budapest.

DR. M. HENNEQUIN has been appointed director of mathematical conferences of the University of Caen.

DR. M. PLANCHEREL, of the University of Geneva, has been appointed associate professor of mathematics at the University of Freiburg, Switzerland.

DR. W. LÁSKA, of the technical school at Lemberg, has been appointed professor of mathematics at the Bohemian University of Prague.

PROFESSOR E. SCHMIDT, of the University of Erlangen, has accepted a professorship of mathematics at the University of Breslau.

PROFESSOR G. CANTOR, of the University of Halle, has been elected a corresponding member of the royal institute of Venice.

DR. E. ZORETTI has been appointed professor of rational mechanics at the University of Caen.

PROFESSOR L. ISELY, of the Swiss University at Neuchâtel, has retired from active teaching.

PROFESSOR F. D'ARCAIS, of the University of Padua, has been elected to membership in the royal institute of Venice.

THE royal academy of sciences of Naples has recently elected Professor D. MONTESANO resident member, Professor U. DINI, of the University of Pisa, non-resident member, and Professor G. GUCCIA, of the University of Palermo, corresponding member.

PROFESSOR R. DEDEKIND, of the technical school at Brunswick, and Professor Sir J. LARMOR, of the University of Cambridge, have been elected foreign members of the royal academy dei Lincei, of Rome; Professor O. TEDONE, of the University of Genoa, has been elected to corresponding membership.

PROFESSOR T. LEVI-CIVITA, of the University of Padua, has been elected corresponding member of the academy of sciences of Paris.

THE honorary degree of doctor of science has been conferred by Colgate University upon Professor H. E. SLAUGHT, of the University of Chicago.

PROFESSOR L. G. WELD, of the University of Iowa, has resigned. Professor A. G. SMITH succeeds Professor Weld as head of the department of mathematics.

AT Dartmouth College, Professor A. D. PITCHER, of the University of Kansas has been appointed assistant professor of mathematics; Mr. MEYER GABA, of the Rolla School of

Mines, and Mr. C. R. DINES, of Harvard University, have been appointed instructors in mathematics.

AT the University of Kansas, Professor J. N. VAN DER VRIES has been elected chairman of the department of mathematics; Dr. H. N. JORDAN has been appointed assistant professor of mathematics; Miss HAZEL MCGREGOR, Mr. J. J. WHEELER, and Mr. J. O. HASSLER have been appointed instructors in mathematics.

DR. H. E. BUCHANAN, of the University of Wisconsin, has accepted the professorship of mathematics at Carleton College, Northfield, Minnesota.

MR. W. E. SMITH has been appointed assistant professor of mathematics at the Pennsylvania State College.

DR. D. BUCHANAN has been appointed assistant professor of mathematics at Queen's College, Kingston, Ontario.

REV. A. S. HAWKESWORTH has resigned his lectureship in higher mathematics and semitic languages in the University of Pittsburgh.

DR. H. W. MOTT has been appointed assistant professor of mathematics at the University of Wisconsin.

DR. R. L. MOORE, of Northwestern University, has been appointed instructor in mathematics at the University of Pennsylvania.

MR. E. P. R. DUVAL has been appointed instructor in mathematics at Princeton University.

AT Middlebury College Mr. G. H. CRESSE has been appointed head of the department of mathematics with the rank of assistant professor.

MR. W. C. KRATHWOHL has been appointed instructor in mathematics in Washington University, St. Louis.

DR. J. E. ROWE will have charge of mathematical work at Haverford College during the sabbatical absence of Professor L. W. Reid.

DR. G. D. GABLE, professor of mathematics at Wooster University, died August 23 at the age of forty-eight. Professor Gable had been a member of the American Mathematical Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARNOUX (G.). Essai de géométrie analytique modulaire a deux dimensions. Paris, Gauthier-Villars, 1911. 8vo. 11+160 pp. Fr. 6.00.

BECKER (G. F.) and VAN ORSTRAND (C. E.). Hyperbolic functions. New edition. (Smithsonian mathematical tables.) Washington, D. C., Smithsonian Institution, 1911. 8vo. 51+321 pp. $4.00

BELTRAMI (E.). Opere matematiche, pubblicate per cura della facoltà di scienze della r. università di Roma. Tomo III. Milano, Hoepli, 1911. 4to. 488 pp. L. 25.00

BIRCKENSTAEDT (M.). Zwei neue allgemeine Differentiationsgesetze. (Progr.) Altona, 1911. 4to. 11 pp.

BLUM (F.). Die infinitesimale Biegung von Flächen bei vollständiger Starrheit eines Kurvensystems. (Diss.) Tübingen, 1909. 8vo. 41 pp.

BRAUN (W.). Bestimmung der Körperdiscriminante in einem kubischen Zahlkörper. (Diss.) Strassburg, 1909. 8vo. 54 pp.

BURGWEDEL (R.). Ueber die Euler'schen und Gauss'schen Methoden der Primzahlbestimmung. (Diss.) Strassburg, 1910. 8vo. 140 pp.

BURNSIDE (W.). Theory of groups of finite order. 2nd edition. London, Cambridge University Press, 1911. 8vo. 536 pp. 15 s.

CLAIRIN (J.). Cours de mathématiques générales. Tome 2: Calcul intégral et applications géométriques. Lille, Janny, 1911. 4to. 156 pp. Fr. 10.00

DENTON (W. W.). See YOUNG (J. W.).

DIENGER (K.). Beitrag zur Lehre von den arithmetischen und geometrischen Reihen höherer Ordnung. (Progr.) Rastatt, 1910. 4to. 16 pp.

DIESING (M.). Einführung in die Differentialrechnung und Anwendung derselben auf Maxima, Minima, unendliche Reihen und Quotienten in der unbestimmten Form $\frac{0}{0}$. (Progr.) Halle, 1911. 4to. 35 pp.

EIBENBAUM (C.). Die Gleichung des Pythagoras mit der Einschränkung des Fermat. Berlin, Schultz, 1911. 4 pp. M. 3.00

ENGELHARDT (P.). Untersuchungen über die im Schlusswort des Lie'schen Werkes "Geometrie der Berührungstransformationen" angedeuteten Probleme. (Diss.) Würzburg, 1910. 8vo. 65 pp.

ENRIQUES (F.). See FRAGEN der Elementargeometrie.

FRAGEN der Elementargeometrie. Aufsätze von U. Amaldi, E. Baroni, R. Bonola und anderen. Zusammengestellt von F. Enriques. 1ter Teil: Die Grundlagen der Geometrie. Deutsche Ausgabe von H. Thieme. Leipzig, Teubner, 1911. 8vo. 10+366 pp. Cloth. M. 10.00

GARNIER (R.). Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. Paris, Gauthier-Villars, 1911. 8vo. 133 pp.

GAU (E.). Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode de M. Darboux. (Thèse.) Paris, Gauthier-Villars, 1911. 4to. 123 pp.

GOERL (W.). Ueber die Spirale am Kegel $x^2+y^2-tg^2d.z^2=0$. 1ter Teil. (Progr.) Leitmeritz, 1910. 8vo. 19 pp.

GRÄBNER (G.). Algebraische Bertrand-kurven und algebraische Kurven constanter Torsion. (Diss.) Würzburg, 1909. 8vo. 73 pp.

GRELLING (K.). Die Axiome der Arithmetik mit besonderer Berücksichtigung der Beziehungen zur Mengenlehre. (Diss.) Göttingen, 1910. 8vo. 26 pp.

GROSS (G.). Zur Kenntnis des Lambertschen Kosinusgesetzes. (Diss.) Breslau, 1911. 8vo. 37 pp.

HECKE (E.). Zur Theorie der Modulfunktionen von zwei Variabeln und ihrer Anwendung auf die Zahlentheorie. (Diss.) Göttingen, 1910. 8vo. 37 pp.

HOLBA (S.). Fermats letzter Satz als Minimumaufgabe. Budapest, Kilián, 1911. 8vo. 29 pp. M. 1.00

HURWITZ (W. A.). Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung. (Diss.) Göttingen, 1910. 8vo. 97 pp.

JANISZEWSKI (S.). Sur les continus irréductibles entre deux points. (Thèse.) Paris, Gauthier-Villars, 1911. 4to. 99 pp.

MCNEILE (A. M.) and (J. D.). A school calculus. London, Murray, 1911. 8vo. 388 pp. 7s. 6d.

MATHÉMATIQUES et mathématiciens. Pensées et curiosités, recueillies par A. Rebière . 4e édition. Paris, Vuibert, 1911. 8vo. 570 pp.

MERCER (J.). Sturm-Liouville series of normal functions in the theory of integral equations. London, Dulau, 1911. 4to. Sewed. 4s.

MESSURIER (T. A. Le). Key to Prof. W. W. Johnson's differential equations. London, Chapman & Hall, 1911. 8vo. 7s. 6d.

MICHELS (P.). Einiges über die Anwendung der ähnlichen Abbildung. (Progr.) Meseritz, 1911. 4to. 27 pp.

MÜLLER (A.). Galileo Galilei: studio storico, scientifico. Traduzione di P. Perciballi, con prefazione di P. Maffi, e lettera di G. Schiaparelli. Roma, Bretschneider, 1911. 8vo. 19+522 pp. L. 10.00

MÜLLER (F.). Der mathematische Sternhimmel des Jahres 1811. Rückblicke auf die Mathematik vor 100 Jahren. Festschrift. Leipzig, Teubner, 1911. 8vo. 29 pp. M. 0.80

MÜLLER (W.). Die rationale Kurve fünfter Ordnung im fünf-, vier-, drei-, und zweidimensionalen Raum. (Diss.) Leipzig, 1910. 8vo. 100 pp.

NEUE BÜCHER über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet. Mitgeteilt Frühjahr, 1911. Leipzig, Hinrichs. 8vo. M. 0.30

NIELSEN (N.). Théorie des fonctions métasphériques. Paris, Gauthier-Villars, 1911. 4to. 7+212 pp. Fr. 12.00

PARTINGDON (J. R.). Higher mathematics for chemical students. London, Methuen, 1911. 8vo. 278 pp. 5 s.

REBIÈRE (A.). See MATHÉMATIQUES et mathématiciens.

RUDOLPHI (W.). Analytische Geometrie des Punktes, der Geraden und der Ebene in Verbindung mit darstellender Geometrie. (Progr.) Neumünster, 1911. 4to. 34 pp.

SCHEFFERS (G.). See SERRET (J. A.).

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnacks Uebersetzung neu bearbeitet von G. Scheffers. 2ter Band: Integralrechnung. Leipzig, Teubner, 1911. 8vo. 14+639 pp. Cloth. M. 13.00

SOMMERVILLE (D. M. Y.). Bibliography of non-Euclidean geometry; including the theory of parallels, the foundations of geometry and space of n dimensions. London, Harrison, 1911. 8vo. 400 pp. 10 s.

VAN ORSTRAND (C. E.). See BECKER (G. F.).

VIVANTI (G.). Lezioni di analisi infinitesimale. Pavia, Mattei, 1911. 8vo. 11+648 pp. L. 15.00

VOTTERO (G.). Un nuovo ellissografo: teoria, descrizione, applicazione. Torino, Cassone, 1911. 8vo. 7 pp.

WITTSACK (P.). Ueber das identische Verschwinden der Hauptgleichungen der Variation vielfacher Integrale. (Diss.) Heidelberg, 1910. 8vo. 30 pp.

YOUNG (J. W.). Lectures on fundamental concepts of algebra and geometry. Prepared for publication with the cooperation of W. W. Denton, with a note on the growth of algebraic symbolism by U. G. Mitchell. New York, Macmillan, 1911. 8vo. 247 pp. $1.60

II. ELEMENTARY MATHEMATICS.

ABHANDLUNGEN über den mathematischen Unterricht in Deutschland. 3ter Band. 1tes Heft, von R. Schimmack. Die Entwicklung der mathematischen Unterrichtsreform in Deutschland. Leipzig, Teubner, 1911. 8vo. 6+146 pp. M. 3.60

AMIOT (A.). Trattato di geometria elementare. Nuova edizione di A. Socci. 8a impressione. Firenze, LeMonnier, 1911. 8vo. 8+402 pp. L. 2.50

BAKER (W.) and BOURNE (A. A.). A new geometry. London, Bell, 1911. 8vo. 268 pp. 2s. 6d.

BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics and geometry. A textbook for elementary students in technical and trade schools. New York, Van Nostrand, 1911. 12mo. $1.50

BAUSCHINGER (J.) und PETERS (J.). Logarithmisch-trigonometrische Tafeln mit 8 Dezimalstellen. 2ter Band. Leipzig, Engelmann, 1911. 8vo. 952 pp. Cloth. M. 37.00

——. Logarithmic-trigonometrical tables, with 8 decimal places. Volume 1: Table of logarithms. New York, Stechert, 1911. 8vo. 15+367 pp. $4.65

——. Logarithmic-trigonometrical tables, with 8 decimal places. Volume 2: Table of the logarithms of the trigonometrical functions. New York, Stechert, 1911. 8vo. 952 pp. $9.25

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Unterstufe. Ausgabe B. 2te Auflage. Leipzig, Teubner, 1911. 8vo. 8+327 pp. M. 2.80

BERKHAM (G.). Aus dem geometrischen Anfangsunterricht. (Progr.) Hamburg, 1911. 8vo. 30 pp.

BERNARDI (G.). Tavole contenenti i doppi, i quadrati, i tripli dei quadrati ed i cubi dei numeri interi da 1 a 1000, ecc. 2a edizione, rifatta. Bologna, Beltrami, 1911. 8vo. 27+25 pp.

BONASSE (H.). Cours de mathématiques générales spécialement écrit pour les physiciens et les ingénieurs. Paris, Delagrave, 1911. 8vo. 650 pp. Fr. 20.00

BOURLET (C.). Précis d'algèbre. Classes de 3e B, 2e et 1re C et D. 6e édition, revue. Paris, Hachette, 1911. 16mo. 440 pp. Fr. 2.50

BOURNE (A. A.). See BAKER (W.).

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales. 10e édition. Paris, Hachette, 1911. 8vo. 176 pp. Fr. 2.50

BREMIKER (H.). Ueber die Behandlung der Ungleichungen im Unterricht. (Progr.) Berlin, 1911. 4to. 23 pp.

BRÜCHER (K.). Anschauung in der Arithmetik. Bamberg, Buchner, 1911. 8vo. 7+41 pp. M. 1.60

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DAVISON (C.). Exercises from algebra for secondary schools. New York, Putnam, 1911. 12mo. 6+320 pp. $1.00

EREDE (G.). Manuale di geometria pratica. 5a edizione, riveduta e corretta. Milano, Hoepli, 1911. 24mo. 16+257 pp. L. 2.00

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FENKNER (H.) und WAGNER (H.). Lehr- und Uebungsbuch der Mathematik für Lyzeen. 1ter Teil. Berlin, Salle, 1911. 8vo. 6+286 pp. Cloth. M. 3.20

FERRARIS (P.). Elementi di geometria. Vol. II. Torino, Paravia, 1911. 16mo. 174 pp. L. 3.00

FRATCHER (W. F.). Instantaneous calculator. Detroit, 1911. 6 pp. $5.00

FREYMANN (K. L.). Praktische Lösungen mathematischer Aufgaben. Frankfurt a. M., Gerheim, 1911. 8vo. 15 pp. M. 1.30

GAZZANIGA (P.). See VERONESE (G.).

GOLDMANN (H.). Der russische Rechenapparat; seine rationale Anwendung in den Schulen, nebst einigen Untersuchungen über den bildenden Wert der Arithmetik. Wien, Pichler, 1911. 8vo. 3+44 pp. M. 0.80

GÖTTING (E.). See BEHRENDSEN (O.).

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GRÉVY (A.). Traité d'algèbre à l'usage des élèves de mathématiques A et B. 5e édition. Paris, Vuibert. 8vo. 502 pp. Fr. 6.00

HAASE (E.). See WILK (H.).

HEINRICH (M.). Vereinfachter Gang des Anfangsunterrichts in der Planimetrie, analytischen Geometrie und Trigonometrie. Leipzig, Quelle und Meyer, 1911. 8vo. 46 pp. M. 0.80

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. 1tes Heft. 4te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1911. 8vo. 6+104 pp. Cloth. M. 2.40

KRIMPHOFF (W.). See SCHWERING (R.).

KÜHTMANN'S Rechentafeln. Ein handliches Zahlenwerk mit 2 Millionen Lösungen, die alles Multiplizieren und Dividieren ersparen. Dresden, Kühtmann, 1911. 8vo. 476 pp. Cloth. M. 18.00

LISBOA (J. I. de A.). Lições de algebra elementar. Primeiro volume. Paris, Gauthier-Villars, 1911. 8vo. 6+378 pp.

MCCORMACK (T. J.). Why do we study mathematics: a philosophical and historical retrospect. Cedar Rapids, Ia., Torch Press, 1911. 8vo. 26 pp. $0.25

MARTINI (Z. A.). Guida pratica per la risoluzione delle equazioni di 1° e di 2° grado. 3a edizione. Livorno, Giusti, 1911. 16mo. 10+142 pp. L. 1.00

MÜLLER (H.). Koordinatenbegriff und Kegelschnittlehre. Metz, Scriba, 1911. 8vo. 4+79 pp. M. 1.20

MYERS (G. W.) and others. First year mathematics for secondary schools. (University of Chicago mathematical series, edited by E. H. Moore.) Chicago, University of Chicago, 1911. 12mo. 12+365 pp. $1.00

——. Teacher's manual for first-year mathematics. (University of Chicago mathematical series, edited by E. H. Moore.) Chicago, University of Chicago, 1911. 12mo. 9+164 pp. $0.80

NIEWENGLOWSKI (B.). Troisième année de géométrie. Premier cycle. Classe de 3e B. Paris, Delagrave, 1911. 18mo. 195 pp. Fr. 2.75

OESTERLE (F. K.). Wesen und Darstellung der Funktion. Strassburg, Heitz, 1911. 8vo. 19 pp. M. 0.60

OTTO (F.) und SIEMON (P.). Uebungsbuch der Geometrie für höhere Mädchenschulen. 3te Auflage. Leipzig, Hirt, 1911. 8vo. 152 pp. M. 0.60

PAUL (M. O.). Mathematisches Lehr- und Uebungsbuch für höhere Mädchenschulen. 1ter Band: Arithmetik und Algebra. Leipzig, Quelle und Meyer, 1911. 8vo. 4+220 pp. Cloth. M. 2.00

PETERS (J.). Logarithmic tables to seven places of decimals of the trigonometrical functions. Stereotype edition. New York, Stechert, 1911. 4to. 7+921 pp. $10.00

——. Siebenstellige Logarithmentafel der trigonometrischen Funktionen. Leipzig, Engelmann, 1911. 8vo. 8+921 pp. Cloth. M. 30.00

——. See BAUSCHINGER (J.).

RATINET (A.). See BOUVART (C.).

SANDERS (A.). Key to Sanders' plane and solid geometry. New York, American Book Co., 1911. 12mo. 144 pp. Half leather. $1.00

SCHIMMACK (R.). See ABHANDLUNGEN, etc.

SCHWERING (K.) und KRIMPHOFF (W.). Ebene Geometrie. 7te Auflage. Freiburg, Herder, 1911. 8vo. 8+142 pp. Cloth. M. 2.30

SEYBOTH (J.). Tables logarithmiques et trigonométriques à quatres décimales. 2e édition. Paris, Challamel, 1911. 4to. 36 pp.

SIEMON (P.). See OTTO (F.).

SMITH (D. E.). The teaching of geometry. Boston, Ginn, 1911. 12mo. 5+329 pp. $1.25

THOM (A.). Graduated questions in arithmetic, algebra, geometry, trigonometry, analytical geometry, with solutions. London, Gibson, 1911. 12mo. 68 pp. Sewed. 6d.

TREUTLEIN (P.). Der geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichtes. Mit einem Einführungswort von F. Klein. Leipzig, Teubner, 1911. 8vo. 10+216 pp. Cloth. M. 5.60

VERONESE (G.). Elementi di geometria, trattati con la collaborazione di P. Gazzaniga. Parte II. Edizione quarta. Padova, Drucker, 1911. 8vo. 14+245 pp. L. 2.50

WAGNER (H.). See FENKNER (H.).

WARREN (A. T.). Experimental and theoretical course in geometry. 4th edition, revised. London, Clarendon Press, 1911. 8vo. 302 pp. 2s. 6d.

WHITEHEAD (A. N.). An introduction to mathematics. (Home University Library.) New York, Holt, 1911. 12mo. 256 pp. $0.75

WILK (H.) und HAASE (E.). Geometrie für Mittelschulen. 2te Auflage. Dresden, Bleyl und Kaemmerer, 1911. 8vo. 8+144 pp. M. 2.00

ZICKEROW (G.). Das abgekürzte Rechnen. (Progr.) Rawitsch, 1911. 4to. 14 pp.

III. APPLIED MATHEMATICS.

ARNOULT (J.). Sur le mouvement d'un fil dans l'espace. (Thèse.) Paris, Gauthier-Villars, 1911. 4to. 153 pp.

BÄCKLUND (A. V.). Die in der Mechanik angewandte Variation der Integrationskonstanten als Lie'sche Berührungstransformation betrachtet. Stockholm, 1910. 8vo. 82 pp.

BARLOW (C. W. C.). Magnetism and electricity. (Mathematical physics, volume I.) London, University Tutorial Press, 1911.

BILES (J. H.). The design and construction of ships. Volume 2: Stability, resistance, propulsion, etc. London, Griffin, 1911. 8vo. 438 pp. 25s.

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